HOW TO PROVE THE PRODUCT, QUOTIENT AND CHAIN RULES FOR DERIVATIVE FUNCTIONS USING FIRST PRINCIPLES

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PRODUCT RULE

$$A(x) = f(x)g(x)$$

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

the same as adding zero!

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h}$$

$$= \lim_{h \to 0} f(x+h) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f(x+0) \cdot g'(x) + g(x) \cdot f'(x)$$

$$A'(x) = f(x)g'(x) + g(x)f'(x)$$

QUOTIENT RULE

$$A(x) = \frac{f(x)}{g(x)}$$

$$A'(x) = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{\dfrac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x) \cdot h}$$

$$= \lim_{h \to 0} \frac{1}{g(x+h)\,g(x)} \cdot \lim_{h \to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x+h)}{h}$$

$$= \frac{1}{g(x+0)\,g(x)} \cdot \lim_{h \to 0} \frac{f(x+h)\,g(x) - f(x)\,g(x) + f(x)\,g(x) - f(x)\,g(x+h)}{h}$$

$$= \frac{1}{(g(x))^2} \cdot \lim_{h \to 0} \frac{g(x)\left[f(x+h) - f(x)\right] - f(x)\left[g(x+h) - g(x)\right]}{h}$$

$$= \frac{1}{g(x)^2} \cdot g(x) \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} - f(x) \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \frac{1}{(g(x))^2} \cdot g(x)\,f'(x) - f(x)\,g'(x)$$

$$A'(x) = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{(g(x))^2}$$

① $\dfrac{a \times b}{c \times d} = \dfrac{a}{c} \times \dfrac{b}{d}$

② $\lim_{x \to a} \left[f(x)\,g(x)\right]$

$= \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$

CHAIN RULE

$$A(x) = f(g(x))$$

$$A'(x) = \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{h}$$

multiplying by 1

$$= \lim_{h \to 0} \left(\frac{f(g(x+h)) - f(g(x))}{h} \times \frac{g(x+h) - g(x)}{g(x+h) - g(x)} \right)$$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$\frac{a}{b} \times \frac{c}{d} = \frac{a}{d} \times \frac{b}{c}$

$$= \lim_{h \to 0} \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot g'(x)$$

Let $g(x+h) - g(x) = k$, and observe that

(1) as $h \to 0$, $k \to 0$ $\{g(x+0) - g(x) = 0\}$

(2) $g(x+h) = g(x) + k$ {re-arranging}

$$A'(x) = \lim_{k \to 0} \frac{f(g(x)+k) - f(g(x))}{k} \cdot g'(x)$$

* Replace $g(x)$ with 'x', and k with 'h'!

$$= f'(g(x)) \cdot g'(x)$$