PROPOSED SOLUTIONS TO EXERCISES ON THE BASIC PROPERTIES OF NUMBERS

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Exercises from *Calculus, Third Edition (2006) Michael Spivak*

EXERCISE 1

Prove that

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$$

$$x^n - y^n = x^n + (x^{n-1}y) - (x^{n-1}y) + \dots + (x^2y^{n-2}) - (x^2y^{n-2})$$

$$\underbrace{+ (xy^{n-1}) - (xy^{n-1})}_{\text{these all equal zero! Using to factorise}} - y^n$$

separating the positive and negative terms

$$= \left[x^n + x^{n-1}y + \dots + x^2y^{n-2} + xy^{n-1}\right] + \left[-x^{n-1}y - x^{n-2}y^2 - \dots - xy^{n-1} - y^n\right]$$

factoring 'x' from the positive terms, and '-y' from the negative terms.

$$= x(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$$

$$-y(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$$

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \dots + xy^{n-2} + y^{n-1})$$

EXERCISE 2

Prove the following:

(i) $|xy| = |x| \cdot |y|$

$|a| = \sqrt{a^2}$

$$|xy| = \sqrt{(xy)^2} = \sqrt{x^2y^2} = \sqrt{x^2} \cdot \sqrt{y^2}$$
$$= |x| \cdot |y|$$

ii) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, if $x \neq 0$

$$\left|\frac{1}{x}\right| = \sqrt{\left(\frac{1}{x}\right)^2} = \sqrt{\frac{1^2}{x^2}} = \sqrt{1 \cdot x^{-2}}$$
$$= \sqrt{x^{-2}} = (x^2)^{-\frac{1}{2}} = \frac{1}{\sqrt{x^2}} = \frac{1}{|x|}$$

iii) $\frac{|x|}{|y|} = \left|\frac{x}{y}\right|$, if $y \neq 0$

$$\frac{|x|}{|y|} = \frac{\sqrt{x^2}}{\sqrt{y^2}} = \sqrt{\frac{x^2}{y^2}} = \sqrt{\left(\frac{x}{y}\right)^2} = \left|\frac{x}{y}\right|$$

$|a| = \sqrt{a^2}$ $\quad$ $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$ $\quad$ $\frac{a^2}{b^2} = \left(\frac{a}{b}\right)^2$ $\quad$ $|a| = \sqrt{a^2}$

Prove the following:

(iv) $|x-y| \leq |x| + |y|$

$$(x-y)^2 \leq (x+y)^2$$

$$\sqrt{(x-y)^2} \leq \sqrt{(x+y)^2}$$

$$|x-y| \leq |x+y| \leq |x| + |y|$$

$$|x-y| \leq |x| + |y|$$

* Proof that $|x+y| \leq |x| + |y|$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$= |x|^2 + 2xy + |y|^2$$

$$(x+y)^2 \leq |x|^2 + 2 \cdot |x| \cdot |y| + |y|^2$$

$$(x+y)^2 \leq (|x| + |y|)^2$$

$$\sqrt{(x+y)^2} \leq \sqrt{(|x| + |y|)^2}$$

$$|x+y| \leq |(|x| + |y|)|$$

$$|x+y| \leq |x| + |y|$$

Prove the following:

(v) $|x| - |y| \leq |x - y|$

$$(x-y)^2 = x^2 - 2xy + y^2$$

$$(x-y)^2 \geq |x|^2 - 2 \cdot |x| \cdot |y| + |y|^2$$

$$(x-y)^2 \geq (|x| - |y|)^2$$

$$\sqrt{(|x|-|y|)^2} \leq \sqrt{(x-y)^2}$$

$$|(|x| - |y|)| \leq |x-y| \quad (vi)$$

$$|x| - |y| \leq |(|x| - |y|)| \leq |x-y|$$

$$|x| - |y| \leq |x-y|$$

(vii) $|x + y + z| \leq |x| + |y| + |z|$

* indicate when equality holds

$$(x+y+z)^2 = (x+y+z)(x+y+z)$$

$$= x^2 + xy + xz + y^2 + xy + yz$$
$$+ z^2 + xz + yz$$

$$= x^2 + y^2 + z^2 + 2xy + 2yz + 2xz$$

$$(x+y+z)^2 = |x|^2 + |y|^2 + |z|^2 + 2(xy + yz + xz)$$

$$(x+y+z)^2 \leq |x|^2 + |y|^2 + |z|^2 + 2(|x| \cdot |y| + |y| \cdot |z| + |x| \cdot |z|)$$

$$(x+y+z)^2 \leq (|x| + |y| + |z|)^2$$

$$\sqrt{(x+y+z)^2} \leq \sqrt{(|x| + |y| + |z|)^2}$$

$$|x+y+z| \leq |(|x| + |y| + |z|)| = |x| + |y| + |z|$$

$|x+y+z| = |x| + |y| + |z|$ when $x, y, z \geq 0$ OR $x, y, z \leq 0$
OR $xy \geq 0, z = 0$ OR $yz \geq 0, x = 0$
OR $zx \geq 0, y = 0$ $\{\text{for } x \in \mathbb{R}\}$

EXERCISE 3

The maximum of two numbers x and y is denoted by $\max(x,y)$.

Thus, $\max(-1,3) = \max(3,3) = 3$ and $\max(-1,-4)$

and $\max(-4,-1) = -1$.

The minimum of x and y is denoted by $\min(x,y)$.

Prove that

$$\max(x,y) = \frac{x+y+|y-x|}{2},$$

$$\min(x,y) = \frac{x+y-|y-x|}{2}.$$

Derive a formula for $\max(x,y,z)$ and $\min(x,y,z)$

using, for example $\max(x,y,z) = \max(x, \max(y,z))$

* Geometric intuition:

$\frac{x+y}{2} \rightarrow$ midpoint between x & y (e.g. $2\ 3 \overset{3.5}{\downarrow} 4\ 5$ $\quad \frac{2+5}{2} = 3.5$)

$|y-x| \rightarrow$ absolute distance between x and y (e.g. $\overset{1\ \ 2\ \ 3}{2\ 3\ 4\ 5}$ $\quad |2-5| = 3$)

$\frac{|y-x|}{2} \rightarrow$ half of distance between x and y (e.g. $\overset{1.5}{2\ 3\ 4\ 5}$ $\quad \frac{|2-5|}{2} = 1.5$)

$\frac{x+y + |y-x|}{2} \rightarrow$ add half distance to midpoint to find maximum. ($2\ 3 \overset{3.5}{\uparrow} \overset{+1.5}{4\ 5}$ $\quad \frac{2+5+|2-5|}{2} = 5$)

$$\frac{x+y-|y-x|}{2}$$ $\rightarrow$ subtract half distance from midpoint to find minimum.

* Algebraic proof - $\max(x,y)$

1. $|x| = \max(x, -x)$
2. $\max(x,y) + z = \max(x+z,\ y+z)$
3. $z \cdot \max(x,y) = \max(xz,\ yz)$, $z \geqslant 0$

$$|y-x| = \max(y-x,\ x-y) \quad (1)$$

$$|y-x| + x + y = \max(y-x,\ x-y) + (x+y)$$

$$x+y+|y-x| = \max(y-x+(x+y),\ x-y+(x+y)) \quad (2)$$

$$= \max(2y,\ 2x)$$

$$= 2\max(x,y) \quad (3)$$

$$\frac{x+y+|y-x|}{2} = \max(x,y)$$

* Algebraic proof – $\min(x, y)$

$$-|y-x| = \min(y-x,\ x-y)$$

$$+(x+y) - |y-x| = \min(y-x,\ x-y) + (x+y)$$

$$x+y - |y-x| = \min(y-x+(x+y),\ x-y+(x+y))$$

$$= \min(2y,\ 2x)$$

$$x+y-|y-x| = 2\min(x, y)$$

$$\frac{x+y-|y-x|}{2} = \min(x, y)$$

$\therefore$

Deriving a formula for $\max(x,y,z)$

$$\max(x,y,z) = \max\left(x,\ \max(y,z)\right)$$

$$= \max\left(x,\ \left(\frac{y+z+|z-y|}{2}\right)\right)$$

$$= \frac{x + \left(\frac{y+z+|z-y|}{2}\right) + \left|\left(\frac{y+z+|z-y|}{2}\right) - x\right|}{2}$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

$$= \frac{\left(\frac{2x+y+z+|z-y|}{2}\right)}{2} + \frac{\left(\frac{|y+z+|z-y|-2x|}{2}\right)}{2}$$

$$= \frac{2x+y+z+|z-y|}{4} + \frac{|y+z-2x+|z-y||}{4}$$

$$\max(x,y,z) = \frac{2x+y+z+|z-y| + |y+z-2x+|z-y||}{4}$$

Check

$$\max(-1,3,6) = \frac{-2+3+6+|6-3| + |3+6+2+|6-3||}{4} = \frac{10+|11+3|}{4}$$

$$= \frac{24}{4} = 6 \checkmark$$