PROPOSED SOLUTIONS TO EXERCISES ON u-SUBSTITUTION ON INDEFINITE INTEGRALS

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Tolentino Tuition, Calculus I

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CONTENTS

Exercises from [Paul's Online Notes]

PROBLEMS 1 & 2

1. $\int (8x - 12)\left(4x^2 - 12x\right)^4 dx$

$u = 4x^2 - 12x, \quad du = 8x - 12\,dx, \quad dx = 1$

$\int du \cdot u^4 = \int u^4\,du$

$= \frac{1}{5}u^5 + c$

$= \frac{1}{5}(4x^2 - 12x)^5 + c$

2. $\int 3t^{-4}\left(2 + 4t^{-3}\right)^{-7} dt$

$u = 2 + 4t^{-3}, \quad du = -12t^{-4}\,dt$

$(-12t^{-4}dt)/4 = 3t^{-4}dt = -\frac{1}{4}du$

$\int -\frac{1}{4}du \cdot u^{-7} = -\frac{1}{4}\int u^{-7}\,du$

$= -\frac{1}{4}\left(-\frac{1}{6}u^{-6} + c\right)$

$= \frac{1}{24}u^{-6} + c$

$= \frac{1}{24}(2 + 4t^{-3})^{-6} + c$

3. $\displaystyle\int (3-4w)\left(4w^2-6w+7\right)^{10}\,dw$

$u = 4w^2 - 6w + 7 \quad , \quad du = 8w - 6\,dw$

$(8w - 6\,dw) \cdot -\frac{1}{2} = 3 - 4w\,dw = -\frac{1}{2}\,du$

$\int -\frac{1}{2}du \cdot u^{10} = -\frac{1}{2}\int u^{10}\,du$

$= -\frac{1}{2} \cdot \frac{1}{11} u^{11} + c$

$= -\frac{1}{22}(4w^2 - 6w + 7)^{11} + c$

4. $\displaystyle\int 5\,(z-4)\;\sqrt[3]{z^2-8z}\,dz$

$u = z^2 - 8z \quad , \quad du = 2z - 8\,dz$

$(2z - 8\,dz) \cdot \frac{5}{2} = 5z - 20\,dz = 5(z-4)\,dz = \frac{5}{2}\,du$

$\int \frac{5}{2}du \cdot u^{\frac{1}{3}} = \frac{5}{2}\int u^{\frac{1}{3}}\,du$

$= \frac{5}{2} \cdot \frac{3}{4} u^{\frac{4}{3}} + c$

$= \frac{15}{8}(z^2 - 8z)^{\frac{4}{3}} + c$

5. $\displaystyle\int 90x^2 \sin\left(2+6x^3\right)\,dx$

$u = 2+6x^3 \quad , \quad du = 18x^2\,dx$

$(18x^2\,dx)\cdot 5 = 90x^2\,dx = 5\,du$

$$\int 5\,du \cdot \sin(u) = 5\int \sin(u)\,du$$

$$= 5\cdot -\cos(u) + C$$

$$= -5\cos\left(2+6x^3\right) + C$$

6. $\displaystyle\int \sec(1-z)\tan(1-z)\,dz$

$u = 1-z \quad , \quad du = -1\,dz$

$(-1\,dz)\cdot -1 = dz = -du$

$$-\int \sec(u)\tan(u)\,du = \boxed{-\sec(u)+C}^{\,*}$$

$$= -\sec(1-z)+C$$

$$\int \sec(x)\tan(x)\,dx = \int \frac{1}{\cos(x)}\cdot\frac{\sin(x)}{\cos(x)}\,dx$$

$$= \int \frac{\sin(x)}{\cos^2(x)}\,dx \qquad \left\{\begin{array}{l} u = \cos(x) \\ du = -\sin(x) \end{array}\right\}$$

$$= \int \frac{-du}{u^2}$$

$$= -\int \frac{1}{u^2}\,du$$

$$= -\left(-\frac{1}{u}\right)+C$$

$$\int \sec(x)\tan(x)\,dx = \frac{1}{\cos(x)}+C = \sec(x)+C$$

7. $\int \left(15t^{-2} - 5t\right) \cos\left(6t^{-1} + t^2\right)\, dt$

$u = 6t^{-1} + t^2 \quad , \quad du = -6t^{-2} + 2t\, dt$

$(-6t^{-2} + 2t\,dt) \cdot -\frac{15}{6} = 15t^{-2} - 5t\,dt = -\frac{15}{6}\,du = -\frac{5}{2}\,du$

$\int -\frac{5}{2}\,du \cdot \cos(u) = -\frac{5}{2} \int \cos(u)\,du$

$= -\frac{5}{2} \sin(u) + c$

$= -\frac{5}{2} \sin(6t^{-1} + t^2) + c$

8. $\int \left(7y - 2y^3\right) e^{y^4 - 7y^2}\, dy$

$u = y^4 - 7y^2 \quad , \quad du = 4y^3 - 14y\,dy$

$(4y^3 - 14y\,dy) \cdot -\frac{1}{2} = 7y - 2y^3\,dy = -\frac{1}{2}\,du$

$\int -\frac{1}{2}\,du \cdot e^u = -\frac{1}{2} \int e^u\,du$

$= -\frac{1}{2} e^u + c$

$= -\frac{1}{2} e^{y^4 - 7y^2} + C$

9. $\displaystyle\int \frac{4w+3}{4w^2+6w-1}\,dw$

$u = 4w^2 - 6w - 1 \quad , \quad du = 8w - 6\ dw$

$(8w-6\ dw)\cdot\frac{1}{2} = 4w+3\ dw = \frac{1}{2}\,du$

$$\int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\cdot \ln|u| + c$$

$$= \frac{1}{2}\ln|4w^2-6w-1| + c$$

10. $\displaystyle\int \left(\cos(3t) - t^2\right)\left(\sin(3t) - t^3\right)^5 dt$

$u = \sin(3t) - t^3 \quad , \quad du = 3\cos(3t) - 3t^2\ dt$

$(3\cos(3t) - 3t^2\ dt)\cdot\frac{1}{3} = \cos(3t) - t^2\ dt = \frac{1}{3}\,du$

$$\int \frac{1}{3}\,du \cdot u^5 = \frac{1}{3}\int u^5\,du$$

$$= \frac{1}{3}\cdot\frac{1}{6}u^6 + c$$

$$= \frac{1}{18}\left(\sin(3t) - t^3\right)^6 + c$$

11. $\int 4\left(\frac{1}{z} - \mathbf{e}^{-z}\right) \cos\left(\mathbf{e}^{-z} + \ln z\right)\, dz$

$u = e^{-z} + \ln z$, $du = -e^{-z} + \frac{1}{z}\, dz$

$\int 4\, du \cdot \cos(u) = 4 \int \cos(u)\, du$

$= 4 \sin(u) + c$

$= 4(\sin(e^{-z} + \ln z) + c$

12. $\int \sec^2(v)\, \mathbf{e}^{1+\tan(v)}\, dv$

$u = 1 + \tan(v)$

$du = \sec^2(v)\, dv$

$\frac{d}{dv} \tan(v) = \frac{d}{dv} \frac{\sin(v)}{\cos(v)}$

$= \frac{\cos(v) \cdot \cos(v) - \sin(v) \cdot - \sin(v)}{\cos^2(v)}$

$= \frac{\cos^2(v) + \sin^2(v)}{\cos^2(v)}$

$= \frac{1}{\cos^2(v)}$

$= \sec^2(v)$

$\int e^u\, du$

$= e^u + c$

$= e^{1 + \tan(v)} + c$

PROBLEMS 13 & 14

13. $\int 10\sin(2x)\cos(2x)\sqrt{\cos^2(2x)+5}\,dx$

$u = \cos^2(2x)+5 = \cos(2x)\cdot\cos(2x)+5$

$du = -2\sin(2x)\cdot\cos(2x) + \cos(2x)\cdot -2\sin(2x)\ dx$

$= -4\sin(2x)\cos(2x)\ dx$

$(-4\sin(2x)\cos(2x)\,dx)\cdot -\frac{10}{4} = 10\sin(2x)\cos(2x)\,dx = -\frac{10}{4}\,du$

$\int -\frac{10}{4}\,du\cdot\sqrt{u} = -\frac{10}{4}\int\sqrt{u}\,du$

$= -\frac{10}{4}\cdot\frac{2}{3}u^{\frac{3}{2}} + c$

$= -\frac{5}{3}\left(\cos^2(2x)+5\right)^{\frac{3}{2}} + c$

14. $\int \frac{\csc(x)\cot(x)}{2-\csc(x)}\,dx$

$u = 2-\csc(x) = 2-\frac{1}{\sin(x)}$, $du = -\left(\frac{0-\cos(x)}{\sin^2(x)}\right)dx$

$= \frac{\cos(x)}{\sin^2(x)}\,dx$

$= \frac{1}{\sin(x)}\cdot\frac{\cos(x)}{\sin(x)}\,dx$

$du = \csc(x)\cot(x)\,dx$

$\int\frac{du}{u} = \int\frac{1}{u}\,du$

$= \ln\left|2-\csc(x)\right| + c$

☆ **15.** $\int \frac{6}{7+y^2}\,dy = \int \frac{6}{7(1+\frac{1}{7}y^2)}\,dy = \frac{6}{7}\int \frac{1}{1+\frac{1}{7}y^2}\,dy$

Because $\frac{d}{dy}\arctan(y) = \frac{1}{1+y^2}$ $\quad = \frac{6}{7}\int \frac{1}{1+\left(\frac{y}{\sqrt{7}}\right)^2}\,dy$

$u = \frac{y}{\sqrt{7}}, \quad du = \frac{1}{\sqrt{7}}\,dy$

$\left(\frac{1}{\sqrt{7}}\,dy\right)\cdot\frac{6}{\sqrt{7}} = \frac{6}{7}\,dy \quad \frac{6}{\sqrt{7}}\,du$

$\frac{6}{\sqrt{7}}\,du \int \frac{1}{1+u^2} = \frac{6}{\sqrt{7}}\int \frac{1}{1+u^2}\,du$

$= \frac{6}{\sqrt{7}}\arctan\left(\frac{y}{\sqrt{7}}\right) + C$

☆ **16.** $\int \frac{1}{\sqrt{4-9w^2}}\,dw = \int \frac{1}{\sqrt{4(1-\frac{9}{4}w^2)}}\,dw = \frac{1}{2}\int \frac{1}{\sqrt{1-\left(\frac{3}{2}w\right)^2}}\,dw$

Because $\frac{d}{dw}\arcsin(w) = \frac{1}{\sqrt{1-w^2}}$

$u = \frac{3}{2}w, \quad du = \frac{3}{2}\,dw$

$\left(\frac{3}{2}\,dw\right)\cdot\frac{1}{3} = \frac{1}{2}\,dw = \frac{1}{3}\,du$

$\frac{1}{3}\int \frac{1}{\sqrt{1-u^2}}\,du = \frac{1}{3}\arcsin\left(\frac{3}{2}w\right) + C$

PROBLEM 17

17. Evaluate each of the following integrals.

(a) $\displaystyle\int \frac{3x}{1+9x^2}\,dx$

(b) $\displaystyle\int \frac{3x}{(1+9x^2)^4}\,dx$

(c) $\displaystyle\int \frac{3}{1+9x^2}\,dx$

a) $u = 1 + 9x^2, \quad du = 18x\,dx$

$\cancel{\dfrac{18x\,dx}{6}} = 3x\,dx = \dfrac{1}{6}du$

$$\frac{1}{6}\int \frac{1}{u}\,du = \frac{1}{6}\ln|1+9x^2| + c$$

b) $u = 1 + 9x^2, \quad du = 18x\,dx$

$3x\,dx = \dfrac{1}{6}du$

$$\frac{1}{6}\int \frac{1}{u^4}\,du = \frac{1}{6}\int u^{-4}\,du$$

$$= \frac{1}{6}\cdot -\frac{1}{3}u^{-3} + c$$

$$= -\frac{1}{18}(1+9x^2)^{-3} + c$$

c) $$\int \frac{3}{1+9x^2}\,dx = 3\int \frac{1}{1+(3x)^2}\,dx$$

$u = 3x, \quad du = 3\,dx$

$$\int \frac{1}{1+u^2}\,du = \arctan(u) + c$$

$$= \arctan(3x) + c$$