PROPOSED SOLUTIONS TO EXERCISES ON UNIT VECTORS

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CONTENTS

Exercises from *Sears and Zemansky's University Physics (with Modern Physics) 12th Edition*, 2008

EXERCISE 1.48

1.48 Find the x- and y- components of the vector $\vec{A}$, where

$$\vec{A} = 5.0\vec{B}\ , and\ \vec{B} = 4\hat{\imath} - 6\hat{\jmath}$$

$\boxed{1.48}$

$$\vec{A} = 5.0\vec{B} = \begin{bmatrix} 5.0 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = \begin{bmatrix} 20 \\ -30 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} = 20\hat{i} - 30\hat{j}$$

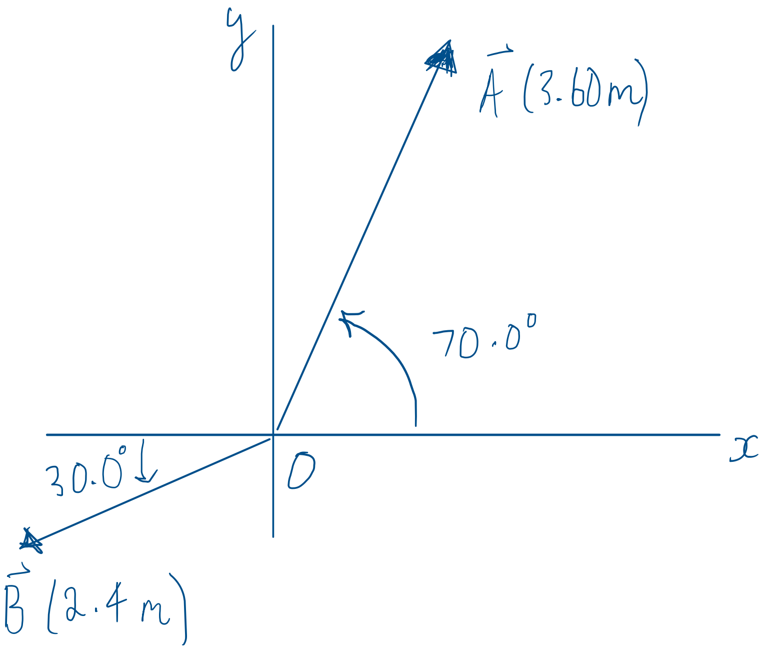
$$= A_x\hat{i} + A_y\hat{j}$$

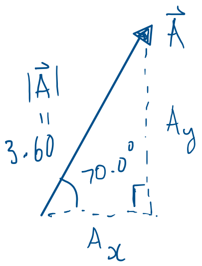
$A_x = 20$, $A_y = -30$

EXERCISE 1.49

1.48 (a) Write each vector in Fig. 137 in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

Figure 1.37



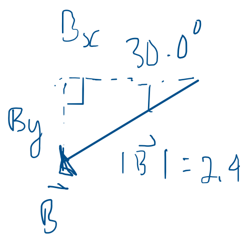


$\cos 70° = \dfrac{A_x}{|\vec{A}|} = \dfrac{A_x}{3.60}$

$A_x = 3.60 \cos 70° \approx 1.23$ m

$\sin 70° = \dfrac{A_y}{|\vec{A}|} = \dfrac{A_y}{3.60}$

$A_y = 3.60 \sin 70° \approx 3.38$ m



$\cos(180+30)° = \dfrac{B_x}{|\vec{B}|}$

$B_x = 2.4 \cos 210° \approx -2.1$m

$\sin 210° = \dfrac{B_y}{2.4}$

$B_y = 2.4 \sin 210° \approx -1.2$m

$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$

$= 1.23\hat{\imath} + 3.38\hat{\jmath}$

$\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath}$

$= -2.1\hat{\imath} - 1.2\hat{\jmath}$

(b) Use unit vectors to express the vector $\vec{C}$, where

$$\vec{C} = 3.00\vec{A} - 4.00\vec{B}$$

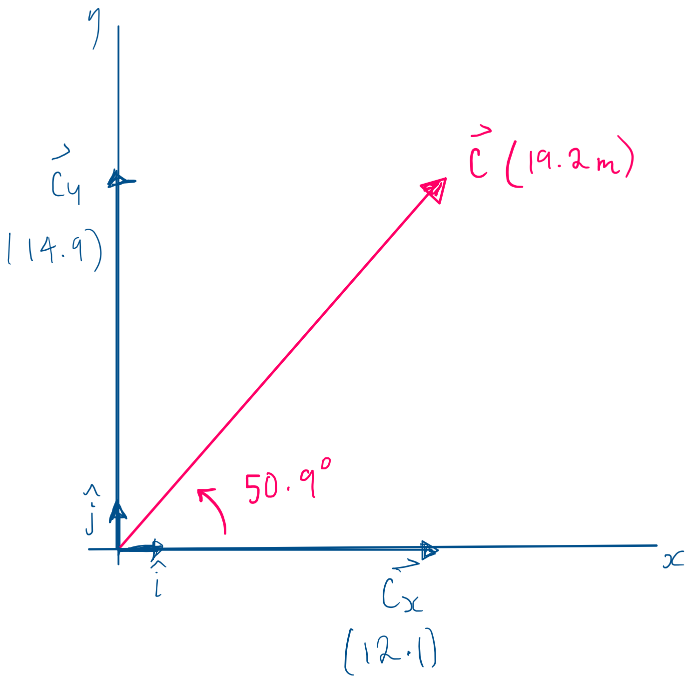
$$\vec{C} = [3.00]\begin{bmatrix} 1.23 \\ 3.38 \end{bmatrix}\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} - [4.00]\begin{bmatrix} -2.1 \\ -1.2 \end{bmatrix}\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}$$

$$= \begin{bmatrix} 3.69 \\ 10.14 \end{bmatrix}\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix} - \begin{bmatrix} -8.4 \\ -4.8 \end{bmatrix}\begin{bmatrix} \hat{i} \\ \hat{j} \end{bmatrix}$$

$$= (3.69 - (-8.4))\,\hat{i} + (10.14 - (-4.8))\,\hat{j}$$

$$\vec{C} = 12.1\hat{i} + 14.9\hat{j}$$

(c) Find the magnitude and direction of $\vec{C}$



$$|\vec{C}| = \sqrt{(C_x)^2 + (C_y)^2}$$

$$= \sqrt{(12.1)^2 + (14.9)^2} \approx 19.2\text{ m}$$

$$\cos\phi = \frac{C_x}{|\vec{C}|} = \frac{12.1}{19.2}$$

$$\phi = \arccos\left(\frac{12.1}{19.2}\right) \approx 50.9°$$

* The direction of $\vec{C}$ is ~50.9° counterclockwise from the positive x-axis.

EXERCISE 1.50

1.50 Given two vectors $\vec{A} = 4.00\hat{\imath} + 3.00\hat{\jmath}$ and $\vec{B} = 5.00\hat{\imath} - 2.00\hat{\jmath}$,

(a) Find the magnitude of each vector

$A_x = 4.00$, $A_y = 3.00$

$|\vec{A}| = \sqrt{(A_x)^2 + (A_y)^2}$

$= \sqrt{16 + 9} = 5.00$

$B_x = 5.00$, $B_y = -2.00$

$|\vec{B}| = \sqrt{(B_x)^2 + (B_y)^2}$

$= \sqrt{25 + 4} \approx 5.39$

(b) Write an expression for the vector difference $\vec{A} - \vec{B}$ using unit vectors

$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

$-\vec{B} = -5.00\hat{i} - (-2.00\hat{j}) = -5.00\hat{i} + 2.00\hat{j}$

$\vec{A} + (-\vec{B}) = (A_x + (-B_x))\hat{i} + (A_y + (-B_y))\hat{j}$

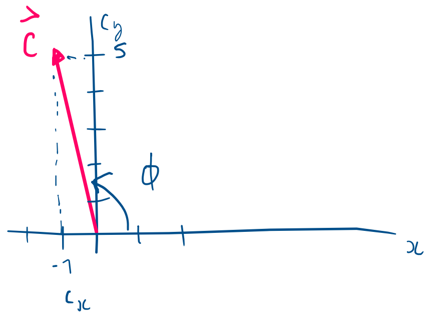
$= (4.00 - 5.00)\hat{i} + (3.00 + 2.00)\hat{j}$

$\vec{A} - \vec{B} = -1.00\hat{i} + 5.00\hat{j}$

(c) Find the magnitude and direction of the vector difference
$\vec{A} - \vec{B}$

Let $\vec{A} - \vec{B} = \vec{C} = -1.00\hat{i} + 5.00\hat{j}$, $C_x = -1.00$ and $C_y = 5.00$

$|\vec{C}| = \sqrt{(C_x)^2 + (C_y)^2} = \sqrt{1+25} \approx 5.10$



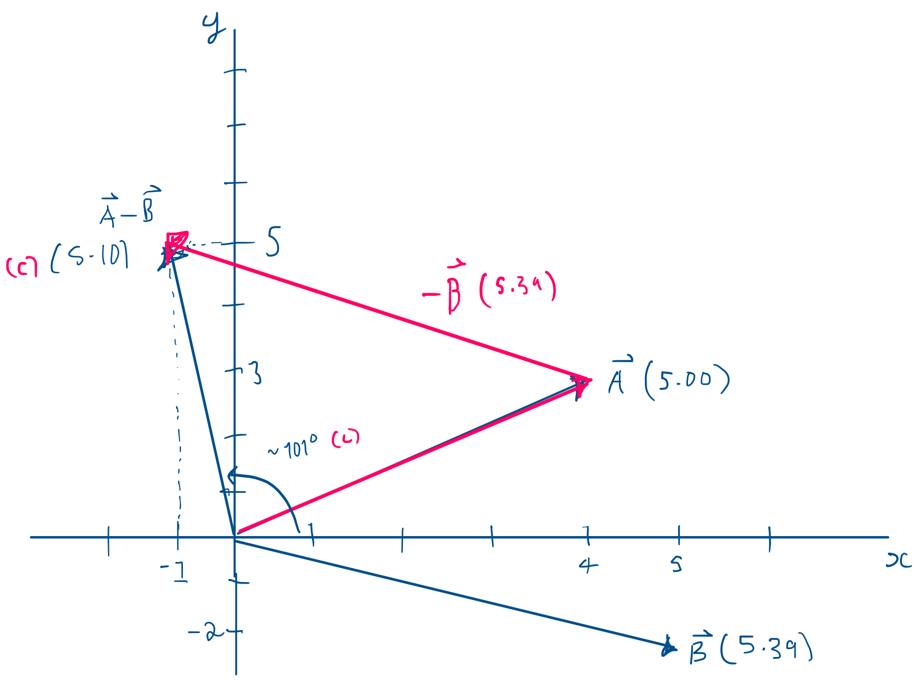
$\sin\left(\phi - \frac{\pi}{2}\right) = \frac{|C_x|}{|\vec{C}|} = \frac{1}{5.10}$

$\phi - \frac{\pi}{2} = \arcsin\left(\frac{1}{5.10}\right)$

$\phi = \frac{\pi}{2} + \arcsin\left(\frac{1}{5.10}\right) \approx 101°$

* The direction of $\vec{A} - \vec{B}$ is 101° counterclockwise from the positive x-axis.

(d) In a vector diagram, show $\vec{A}, \vec{B}$ and $\vec{A} - \vec{B}$ and also show that your diagram agrees qualitatively with your answer in part (c)



EXERCISE 1.51

1.51 (a) Is the vector $(\hat{\imath} + \hat{\jmath} + \hat{k})$ a unit vector? Justify your answer.

$\Downarrow$

Must:

(1) Have a magnitude of 1

(2) Have no units

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad k = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Let $(\hat{i} + \hat{j} + \hat{k}) = \vec{V}$

$$\vec{V} = \begin{bmatrix} 1+0+0 \\ 0+1+0 \\ 0+0+1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{bmatrix} = 1\hat{i} + 1\hat{j} + 1\hat{k}$$

$$V_x = V_y = V_z = 1$$

(1) $$|\vec{V}| = \sqrt{(V_x)^2 + (V_y)^2 + (V_z)^2} = \sqrt{1+1+1} = \sqrt{3} \neq 1$$

A unit vector must have a magnitude equal to 1. $(\hat{i} + \hat{j} + \hat{k})$ has a magnitude of $\sqrt{3}$, which does not equal 1.

$\therefore$ $(\hat{i} + \hat{j} + \hat{k})$ is not a unit vector.

(b)(i) Can a unit vector have any components with a magnitude greater than unity? Justify your answer

$\Downarrow$

'greater than 1'

Let $\vec{V}$ be a unit vector where $|\vec{V}| = 1$, and $\vec{V} \in \mathbb{R}_n$ {$\vec{V}$ has n dimensions}

$|\vec{V}| = \sqrt{(V_1)^2 + (V_2)^2 + \ldots + (V_n)^2} = 1$ {magnitude formula}

Now, let $V_1 > 1$ {let the first component have a magnitude > 1}

Observe that $(V_2)^2, \ldots (V_n)^2 \geq 0$ for all $V_i \in \mathbb{R}$

{if you square any real number, the result will always be either 0 or a positive number – that is, > 0}

And, $(V_1)^2 > 1$ for all $V_1 > 1$

{if you square any real number greater than 1, the result will always be greater than 1}

Therefore $(V_1)^2 + (V_2)^2 + \ldots + (V_n)^2 > 1$, and

{if you add a real number which is greater than 1 to another real number which is either 0 or positive, the result will always be greater than 1}

$\sqrt{(V_1)^2 + (V_2)^2 + \ldots + (V_n)^2} = |\vec{V}| \neq 1$ {only $\sqrt{1} = 1$, but $(V_x)^2 + (V_y)^2 > 1$}

$\therefore$ If a vector $\vec{V}$ has at least 1 component which is greater than 1, its magnitude $|\vec{V}|$ will also be greater than 1. Since a unit vector must have a magnitude of 1, $\vec{V}$ cannot be a unit vector.

(b)(ii) Can a unit vector have any negative components? Justify your answer

$|\vec{V}| = \sqrt{(V_1)^2 + (V_2)^2 + \ldots + (V_n)^2} = 1$, and

$V_1, V_2, \ldots, V_n \leq 1$ (from part b(i))

Now, let $V_1 = -1$,

observe that $(V_2)^2 + \ldots + (V_n)^2 \geq 0$ for all $V_i \in \mathbb{R}$, and

$(V_1)^2 = (-1)^2 = 1$

If $(V_2)^2 + \ldots + (V_n)^2 = 0$, $\left\{ \text{that is, } V_2, \ldots, V_n \text{ are all equal to zero - meaning that } \vec{V} \in \mathbb{R}_1 \text{ (is one-dimensional)} \right\}$

then, $|\vec{V}| = \sqrt{1 + 0} = \sqrt{1} = 1$

$\therefore$ Yes, it is possible for a vector to have a negative component and be a unit vector

(c) If $\vec{A} = a(3.0\hat{\imath} + 4.0\hat{\jmath})$, where a is a constant, determine the value of a that makes $\vec{A}$ a unit vector.

$$\vec{A} = a(3.0\hat{i} + 4.0\hat{j}) = \left[(3.0\,a)\hat{i} + (4.0\,a)\hat{j}\right]$$

$$A_x = 3.0\,a \quad , A_y = 4.0\,a$$

Let $|\vec{A}| = \sqrt{(A_x)^2 + (A_y)^2} = 1 = \sqrt{(3.0a)^2 + (4.0a)^2}$

$$1 = (3.0a)^2 + (4.0a)^2$$

$$= (3.0)^2 a^2 + (4.0)^2 a^2$$

$$= 9a^2 + 16a^2$$

$$1 = 25a^2$$

$$\frac{1}{25} = a^2$$

$$a = \sqrt{\frac{1}{25}} = \frac{\sqrt{1}}{\sqrt{25}} = \frac{1}{5}$$

$\therefore \vec{A} = \frac{1}{5}(3.0\hat{i} + 4.0\hat{j})$ is a unit vector

Check

$$|\vec{A}| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = \sqrt{\frac{9}{25} + \frac{16}{25}} = \sqrt{\frac{25}{25}}$$

$$= \sqrt{1}$$

$$= 1 \checkmark$$